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Which deformations of the Poincaré group?

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Dedicated to Eugene P Wigner

Abstract. We analyse the present status of Poincaré group in considering it as a fundamental object independent of the Minkowski space. We examine the observables associated with it. After having introduced the notion of kinematical observable, we derive the class of 'deformed' Poincaré groups which are compatible with physics.

1. Introduction

The present article is, in a way, the modern version of an earlier one, entitled *Possible Kinematics* (PK), and written 25 years ago [1]. It is modern in that the group theoretical point of view is replaced by an algebraic one. The motivation was the same: finding substitutes for the Poincaré group. However, the present point of view is different from the previous one. In the old work, it was thought that the Poincaré group was responsible for the difficulties of special relativity in particle physics. Since it was realized [2] that it is our notion of space (and space-time) which is not suitable for microphysics‡, the Poincaré group cannot be brought into question, except perhaps at very high energies.

It is not necessary to explain in detail the difficulties contained in classical and quantum special relativity. The interested reader will find information in previous papers [2, 4, 5–10]. However, the present paper is self-contained. It could be entitled *New Possible Kinematics* because, in *contradistinction* to PK, which led its authors to a classification of *already known* kinematics (de Sitter, Poincaré, Galilei) together with some funny approximations as the so-called Carroll kinematics, the present paper proposes essentially a large continuous family of *physically acceptable new* kinematics, among which we find the Poincaré kinematics. The continuity character of this family must be opposed to the group deformations and group contractions involved in PK, which are drastic alternatives of the Poincaré kinematics.

The word *deformation* which appears in the title is already known in the theory of quantum groups to denote a family of Hopf algebras labelled by a parameter q. For $q=0$, the Hopf algebra coincides with the enveloping algebra of a simple (or semi-simple) Lie group. Since the Poincaré group is not semi-simple, it is necessary to find another way to deform it. Three ways have been proposed up to now. One of

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‡ A point which has been underlined by Einstein [3] and Schwinger [4], among other physicists.

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them consists in quantizing space-time [11, 12]. Unfortunately, there is no physical reason to choose a quantum Minkowski space which forces us to reject the isotropy of space together with the standard addition of angular momenta. The same criticism can be made against the q-deformation obtained from a matrix representation of the Poincaré group [13]. An interesting proposal was based on a q-deformation of the complex de Sitter group followed by a contraction [14–16]. Not only this method provides us with an interpretation of the q parameter as a natural unit length but it leaves the rotation subgroup unchanged. Among the possible deformations, Liukerski *et al.* [17] selected one of the deformations they arrived at and studied some of its physical consequences. Other physical investigations followed [18–22].

In the present article, the word deformation will be given a physical signification. The deformation concerns the Poincaré group alone; it means that it does not involve the Minkowski spacetime. A *direct* connection is established with the aid of the observables which appear in its representations. Field theory is ignored.

The paper has seven sections. Section 2 describes the accepted status of the Poincaré group. Section 3 proposes a new status based on a new notion, namely the one of *kinematical observable*†. In this description, the group structure is introduced at the end. It is in releasing this group condition that we are able to derive, in section 4, a physical set of possible deformations of the Poincaré kinematics. The interesting point is that the algebras obtained in this way from a family parametrized by a real number and a function. The Poincaré group belongs to it, the Lukierski–Nowicki–Ruegg quantum Poincaré algebra [15], too‡. The deformed Casimirs are given in section 5. Another section is devoted to the position observable. In the last one, we make some comments and state conclusions.

2. The old and present status of the Poincaré group

It is useful to give, in a new manner, a brief report about the difficulties of special relativity in classical and quantum physics, starting with a few remarks about the ordinary quantization procedure, the one described in all elementary textbooks on quantum mechanics. Quantization is known, in particular, to be a nice trick for introducing the main quantum observables for a *spinless massive* particle, position and momentum, together with their commutation relations. As is well known, we only have to replace the standard Poisson brackets by commutators. Although this procedure was defined for a spinless relativistic particle, its validity was readily accepted for a relativistic spinless massive particle. Concerning *spinning massive* and *massless* particles, this quantization method is *powerless*.

The famous work of Wigner on unitary irreducible representations of the Poincaré group [23] provides us with *some* quantum observables for *all* kinds of particles, including the *massless* and the *spinning* ones. However, there were difficulties in this nice scheme to obtain a correct position observable. The famous Newton–Wigner operator [24] for a spinning particle was suffering many defects. In particular, it does not fit with Minkowski space in that if a particle is localized for an observer, it is not localized for another one. In other words, it is impossible to associate a covariant position in space-time with a localized particle. We give, in table 1 a résumé of the

† This notion is, in a way, a consequence of what I called the de Broglie symmetry principle [4, 9]. It was implicitly used in [10].

‡ It is the work which is at the origin of the present paper.

Table 1. Observables furnished by the quantization procedure and by the Poincaré group Wigner approach.

Particles	*Quantization*	*Poincaré group*
Spinning massive	X, P (S?)	P, S (X?)
Spinning massless	???	P, η (X???)

ability of the two procedures we are spaking about in furnishing kinematical observables. In this table, the letters X, P, S, η denote the position, the momentum, the spin, and the helicity operators, respectively. The interrogation mark is for an unsatisfactory observable, with three interrogation marks in the case where the observable *does not exist.*

In 1966, the author found this situation very unsatisfactory and thought that we had to combine in some way both the quantization and the Poincaré group methods. It was found that *classical* spinning massive particles could be defined from the Poincaré group itself [25, 26]. This idea was investigated extensively by J-M. Souriau [27, 28]. Although proud to have introuced the eight-dimensional phase space for such particles†, the impossibility of defining such a phase space for the photon or the neutrino was disappointing. This difficulty was common to the *usual* classical and quantum mechanical interpretations of the Poincaré group, namely the impossibility of deriving an acceptable position observable. Then, after 1966, the situation was as described in table 2.

From this table, one sees that the non-existence of a position observable for the photon is not purely quantum mechanical. It seems to be attached to the Poincaré group itself‡. In fact, we must underline that the problem of localized states is not *directly* related to the Poincaré group, but to its representations.

If we examine carefully the role of the Poincaré group in physics textbooks, we see that it is always defined from Minkowski space-time or Maxwell's equations but its successes are related to the conservation and additivity of momenta. Even historically, special relativity was recognized as a valid theory because the successes of the formulas

$$E = mc^2 \text{ and } p = \frac{mv}{(1-\beta^2)^{1/2}}$$

better summarized by

$$E^2 - p^2c^2 = m^2c^4 \text{ and } v = \frac{p}{E}.$$

Table 2. Observables furnished by the Poincaré group approach in 'classical' and quantum physics.

Particles	*'Classical'*	*Quantum*
Spinning massive	P, S (X?)	P, S (X?)
Spinning massless	P, η (X???)	P, η (X???))

† It is one of the canonical symplectic manifolds of the Poincaré group (a sphere bundle with the six-dimensional ordinary phase space as a base).
‡ This is the reason for looking for substitutes for the Poincaré group [1, 29].

It is interesting to underline that the way Einstein used to show with the aid of *light* signals, how to measure distances, was purely macroscopic and cannot give any support to a Minkowski space made of points [3, 4, 8].

The Poincaré group is ten-dimensional and its 'ten' momenta are, by definition, the generators of the group, namely the angular momentum $\boldsymbol{J}$, the linear momentum $\boldsymbol{P}$, the energy P_0, and a momentum which has no name, which corresponds to the boosts. We will call this last momentum the *pseudo-angular momentum* $\boldsymbol{K}$.

The momenta have the following properties in a given unitary representation:

(1) Only $\boldsymbol{J}$ is an observable (three self-adjoint operators). The momenta $\boldsymbol{K}$, $\boldsymbol{P}$, and P_0 are represented by unbounded operators, but they will be also referred to as observables.

(2) If the representation describes an isolated system, the momenta (and the symmetrized functions of them) are *conserved* observables. This language implies that we are working in the Heisenberg picture, that is each momentum M_i is considered at time zero, which means that we have to replace, to be precise, M_i by $M_i(0)$. Because P_0 is the generator of time translations, we have

$$M_i(t)=\exp(\mathrm{i}tP_0)M_i(0)\exp(-\mathrm{i}tP_0). \tag{1}$$

It is easy to check that $\boldsymbol{J}$, $\boldsymbol{P}$, and P_0, because they commute with P_0, obey

$$\boldsymbol{J}(t)=\boldsymbol{J}(0) \qquad \boldsymbol{P}(t)=\boldsymbol{P}(0) \qquad P_0(t)=P_0(0). \tag{2}$$

With the pseudo-angular momentum $\boldsymbol{K}$, because

$$[K_i(0), P_0(0)]=\mathrm{i}P_i(0) \tag{3}$$

the conservation law is more subtle; we get

$$\boldsymbol{K}(t)=\exp(\mathrm{i}tP_0)\boldsymbol{K}(0)\exp(-\mathrm{i}tP_0)=\boldsymbol{K}(0)+t\boldsymbol{P}(0). \tag{4}$$

We check that all commutation relations at time t are the same as the ones at time zero. In particular

$$[K_i(t), P_0(t)]=\mathrm{i}P_i(t). \tag{5}$$

The conservation of the pseudo-angular momentum *at time zero* follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{K}(0)&=\frac{\partial}{\partial t}(\boldsymbol{K}(t)-tP_0(t))+\mathrm{i}(P_0(t),\boldsymbol{K}(t)-tP_0(t)]\\ &=-P_0(t)+P_0(t)=0.\end{aligned} \tag{6}$$

For a spinless massive particle, we have

$$\boldsymbol{K}(0)=\tfrac{1}{2}(P_0(0)\boldsymbol{X}(0)+\boldsymbol{X}(0)P_0(0)) \tag{7}$$

where $\boldsymbol{X}$ is the position operator†. It follows that the conservation of $P_0(t)=P_0(0)$ and that of $\boldsymbol{K}(0)=\boldsymbol{K}(t)-t\boldsymbol{P}(t)$ are associated with the conservation of the initial position.

(3) The momenta are additive. The additivity of momenta concerns composite non-interacting systems. In *contradistinction* to conservation, this property characterizes momenta. If an *isolated* system is composed of two non-interacting systems described by the Hilbert spaces $H^{(1)}$ and $H^{(2)}$, the global system is described by the tensor product of the Hilbert spaces and the total momentum M_i is the operator

$$M_i=M_i^{(1)}\otimes\mathrm{Id}^{(2)}\oplus\mathrm{Id}^{(1)}\otimes M_i^{(2)}. \tag{8}$$

† We check that this formula, together with $[X_i(0), P_j(0)]=\mathrm{i}\delta_{ij}$ and $P_0=(\boldsymbol{P}^2+m^2)^{1/2}$ (representation of mass m), is compatible with the commutation relations of the Lie algebra.

We have to keep in mind these three properties in order to propose a rigorous status of the Poincaré group, not from a group theoretical point of view but from a quantum physical one.

3. A new status for the Poincaré group

Let us first make two remarks. First, in quantum physics, we are officially concerned with Hilbert spaces, observables (self-adjoint operators) and canonical transformations (unitary operators). Nevertheless, we used to refer to the momentum operator without saying which system we are interested in. In other words, we speak about an observable *before* introducing any Hilbert space. Second, physicists used to distinguish between *kinematical* groups [1, 29] and *dynamical* groups. As far as quantum mechanics is involved, there is a difference between these two kinds of groups. A kinematical group (the Poincaré group or the Galilei group) is introduced *before* its representations in *contradistinction* to a dynamical group which is defined as a group of unitary transformations, that is in a given representation. In a sense, a kinematical group is an *a priori* group, for which observables could be defined abstractly without referring to self-adjoint operators. That is why we are tempted to introduce a new notion, namely the one of *kinematical observables*. Such a notion is associated with a kinematical group without referring to any of its *representations*.

Another remark is needed. In the standard interpretation of the Poincaré kinematics, we can distinguish between three kinds of observables.

(1) The momenta which have a name independent of the representation.
(2) The momenta without name.
(3) The observables which are defined in a given representation (spin, Newton–Wigner position)

This dissymmetry between observables associated with a kinematical group is not satisfactory. In particular, as we saw earlier, the position observable has a status depending on the representation. That is why we propose to change the status of the Poincaré group with the aid of the following definitions, in which the group structure assumption is put at the end.

Axiom 1. There is an object, called the 'kinematical group', which characterizes the most general isolated system.

Axiom 2. We associate with the *general* isolated system ten *kinematical observables* called *momenta*. They are:

- the angular momentum $\boldsymbol{J}$
- the quasi-angular momentum $\boldsymbol{K}$
- the linear momentum $\boldsymbol{P}$
- the energy P_0.

Let us denote by $\mathscr{A}$ the free algebra generated by these ten kinematical observables. The algebra $\mathscr{A}$ has a unit denoted 1.

Axiom 3. To each momentum M_l corresponds a continuous set of elements of $\mathscr{A}$ called *elementary kinematical elements*. This set is defined by the expression $\exp(-\mathrm{i}\phi M_l)$. We must insist on the fact that, up to now, we introduced none Hilbert space.

Axiom 4. We want to make an elementary kinematical element acting on an arbitrary momentum M_i. For that purpose, we define a mapping μ from $A \times A$ to $\mathscr{A}$ such that

$$\mu(M_i, M_j) = -\mu(M_j, M_i) \in \mathscr{A} \tag{9}$$

and extend this function to $\mathscr{A} \otimes \mathscr{A}$ to $\mathscr{A}$ in imposing the relations

$$\mu(a, b) = -\mu(b, a) \tag{10}$$

$$\mu(a, \mu(b, c)) + \mu(b, \mu(c_k, a)) + \mu(c, \mu(a, b)) = 0 \tag{11}$$

$$\mu(a + b, c) = \mu(a, c) + \mu(b, c) \tag{12}$$

$$\mu(ab, c) = a\mu(b, c) + \mu(a, c)b. \tag{13}$$

This mapping permits definition of a quotient algebra $\mathscr{A}_\mu$ with the aid of the equivalence relation

$$ab - ba \sim \mu(a, b) \tag{14}$$

The algebra is given a *-algebra structure in requiring that the momenta obey $M_i^* = M_i$. An element of $\mathscr{A}_\mu$ is, by definition, a kinematical observable if it is Hermitian $(a = a^*)$.

With the aid of this new algebra, every elementary kinematical element $\exp(-i\phi M_i)$ acts as follows on the momentum M_j

$$M_j \to \exp(-i\phi M_i) M_j \exp(i\phi M_i) = M_j - i\phi[M_i, M_j] - \frac{\phi^2}{2}[M_i, [M_i, M_j]] + \dots \tag{15}$$

where $[M_i, M_j]$ is put for $\mu(M_i, M_j)$.

A kinematical transformation is, by definition, a product of elementary kinematical elements of $\mathscr{A}_\mu$. The kinematical transformations are products of elementary transformations.

The kinematical transformations form a group called the kinematical group. Up to now, it is not necessarily a Lie group.

Axiom 5. Since the kinematical group describes the most general isolated system and that the union of two isolated systems is also an isolated system, there must exist a commutative coproduct Δ in $\mathscr{A}_\mu$. This coproduct must preserve the equivalence relation (14). This is equivalent to impose that $\mathscr{A}_\mu$ must be a bi-algebra.

Axiom 6. A kinematical transformation maps a momentum on a linear combination of momenta. In that case, the kinematical group is a ten-dimensional Lie group and the algebra $\mathscr{A}_\mu$ is nothing else than its enveloping Lie algebra. Moreover, we know that $\mathscr{A}_\mu$ is a bi-algebra with Δ being the mapping

$$\Delta\colon M_i \to (M_i \otimes 1) \oplus (1 \otimes M_i). \tag{16}$$

We also know that $\mathscr{A}_\mu$ is a Hopf algebra with the antipode S defined as the mapping

$$S\colon M_i \to M_i. \tag{17}$$

Axiom 7. The last axiom is trivial: the kinematical group is isomorphic to the Poincaré group.

Our way of defining the Poincaré group is rather strange. The reader would prefer to introduce the enveloping algebra of the Poincaré group as the set of kinematical observables. Nevertheless, our definition is needed in order to replace the Poincaré group by a *physically meaningful* quantum group. This is obtained in replacing axioms 6 and 7 by suitable ones. In this scheme, the algebra $\mathcal{A}_\mu$ becomes the deformed enveloping algebra and the kinematical group becomes a deformed Poincaré group (generally, an infinite-dimensional group).

One of the most important axioms is the fifth one, where the notion of kinematical observable is introduced. It is worthwhile to underline the following fact: if we require the position to be also a kinematical observable, we are immediately led to adopt, for the Poincaré group, the definition

$$X=\frac{1}{2P_0}K+K\frac{1}{2P_0} \tag{18}$$

that is the one proposed in [6].

4. Deformations of the Poncaré group

The deformations we are going to consider are based on the five first axioms of section 3 and the following assumptions.

Assumption 1. The momenta $\boldsymbol{J}$, $\boldsymbol{K}$, and $\boldsymbol{P}$ are vectors and the energy P_0 is a scalar. This means that we have the following commutation relations

$$[J_i, J_j]=i\varepsilon_{ijk}J_k \tag{C1}$$

$$[J_i, P_j]=i\varepsilon_{ijk}P_k \tag{C2}$$

$$[J_i, K_j]=i\varepsilon_{ijk}K_k \tag{C3}$$

$$[J_i, P_0]=0 \tag{C4}$$

Assumption 2. In order to preserve the existence of a relationship between momentum, energy and mass, we impose

$$[P_i, P_j]=0 \tag{C5}$$

$$[P_0, P_i]=0 \tag{C6}$$

We note that the relations (C1), (C2), (C4), (C5), and (C6) define the Lie algebra of the Aristotle group†.

Assumption 3. Now, we require that a boost in the direction i modifies the energy and the P_i component of the momentum. This means that the boosts act—not necessarily linearly—on the energy–momentum space.

$$[K_i, P_0]=\mathrm{i}\alpha(P_0)P_i \tag{C7}$$

$$[K_i, P_j]=\mathrm{i}\beta(P_0)\delta_{ij} \tag{C8}$$

where α and β are two functions to be determined‡. The Jacobi relation, for $i\neq j$

$$[[K_i, P_i], P_j]+[[P_i, P_j]K_i]+[[P_j, K_i]P_i]=0$$

† This group is a seven-dimensional subgroup of the Poincaré group and the Galilei group. The reference to Aristotle is due to Souriau.

‡ It can be shown that the assumption 3 could be replaced by the unique condition that there exists a function $\phi(P_0)$ such that $\phi(P_0)-P^2$ is an invariant observable.

together with (C5), (C7), and (C8) imposes condition (C6). This proves that assumptions 2 and 3 are not independent.

The Jacobi relation

$$[[K_i, K_j], P_k] + [[K_j, P_k]K_i] + [[P_k, K_i], K_j] = 0$$

together with (C8) gives

$$[[K_i, K_j], P_k] = 0 \qquad \text{for } i \neq j \neq k \neq i \tag{19}$$

$$[[K_i, K_j], K_i] = -\mathrm{i}\alpha(P_0)\beta'(P_o)P_j \qquad \text{for } i \neq j$$

Since the commutator $[K_i, K_j]$ is antisymmetric, it is necessarily of the form $\varepsilon_{ijk}V_k$, where $\boldsymbol{V}$ is a vector. This property comes from the well-known rule of angular momenta addition: $D_1 \otimes D_1 = D_0 \oplus D_1 \oplus D_2$. The representations D_0 and D_2 (resp. D_1) belong to the symmetric (resp. antisymmetric) part of the product. Among the three vectors we have $(\boldsymbol{J}, \boldsymbol{K}, \boldsymbol{P})$, only two may be involved, namely $\boldsymbol{J}$ and $\boldsymbol{P}$. This is due to the fact that the Jacobi relation

$$[[K_i, K_j], P_0] + [[K_j, P_0], K_i] + [[P_0, K_i], K_j] = 0$$

implies

$$[[K_i, K_j], P_0] = 0 \tag{20}$$

If we set

$$[K_i, K_j] = \mathrm{i}\varepsilon_{ijk}(\gamma J_k - \delta P_k)$$

the conditions (19) and (20) give

$$[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}\left(\alpha(P_0)\beta'(P_0)J_k - \frac{g(P_0)}{4}(\boldsymbol{J}.\boldsymbol{P})P_k\right). \tag{C9}$$

Finally, the Jacobi relation

$$[[K_i, K_j], K_k] + [[K_j, K_k], K_i] + [[K_k, K_i], K_j] = 0$$

imposes the condition

$$\alpha(P_0)^2\beta''(P_0) + \alpha(P_0)\alpha'(P_0)\beta'(P_0) - \beta(P_0)g(P_0) - \alpha(P_0)g'(P_0)\boldsymbol{P}^2 = 0.$$

Our point of view imposes that this relation is independent of the representation. Therefore, the last term, the one involving $\boldsymbol{P}^2$, must vanish independently. It follows that $g'(P_0)$ vanishes, that is $g(P_0)$ is a constant. We denote this constant by g.

The functions α and β and the constant g obey the relation

$$\alpha(P_0)\frac{\mathrm{d}}{\mathrm{d}P_0}(\alpha(P_0)\beta'(P_0)) - g\beta(P_0) = 0. \tag{21}$$

As a résumé, we have the following commutators for the deformed Poincaré group

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk}J_k \tag{C1}$$

$$[J_i, P_j] = \mathrm{i}\varepsilon_{ijk}P_k \tag{C2}$$

$$[J_i, K_j] = \mathrm{i}\varepsilon_{ijk}K_k \tag{C3}$$

$$[J_i, P_0] = 0 \tag{C4}$$

$$[P_i, P_j] = 0 \tag{C5}$$

$$[P_0, P_i] = 0 \tag{C6}$$

$$[K_i, P_0] = \mathrm{i}\alpha(P_0)P_i \tag{C7}$$

$$[K_i, P_j] = \mathrm{i}\beta(P_0\delta_{ij} \tag{C8}$$

$$[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}\left(\alpha(P_0)\beta'(P_0)J_k - \frac{g}{4}(\boldsymbol{J}.\boldsymbol{P})P_k\right) \tag{C9}$$

where the functions α, β, and the constant g obey equation (21).

Remark. To be called a deformation of the Poincaré group, we must require the conditions $\alpha(P_0) \sim 1$ and $\beta(P_0) \sim P_0$ for small values of P_0. We note that the conditions $\alpha(P_0) \sim 0$ and $\beta(P_0) \sim$ constant would correspond to a deformation of the Galilei group.

We now give a simpler description of the deformations of the Poincaré group. For this purpose, we introduce the following function

$$f(P_0) = \int_0^{P_0} \frac{\mathrm{d}u}{\alpha(u)} \qquad \alpha(P_0) = \frac{1}{f'(P_0)}. \tag{22}$$

If we replace the variable P_0 by $\nu = f(P_0)$, we get, instead of (21)

$$\frac{\mathrm{d}^2\beta(\nu)}{\mathrm{d}\nu^2} = g\beta(\nu)$$

which gives, for $g \neq 0$, solutions of the type

$$\beta(P_0) = A\exp(\sqrt{g}f(P_0)) + B\exp(-\sqrt{g}f(P_0)).$$

The fact that β is equivalent to P_0 implies that we have

$$\beta(P_0) = \frac{\sinh(\sqrt{g}f(P_0))}{\sqrt{g}} \tag{23}$$

Consequently

$$\alpha(P_0)\beta'(P_0) = \cosh(\sqrt{g}f(P_0)).$$

As a conclusion, the three equations involved in the deformations can be replaced by

$$[K_i, P_0] = \mathrm{i}\frac{1}{f'(P_0)}P_i \tag{C7$'$}$$

$$[K_i, P_j] = \mathrm{i}\frac{\sinh\sqrt{g}f(P_0))}{\sqrt{g}}\delta_{ij} \tag{C8$'$}$$

$$[K_i, K_j] = -\mathrm{i}\varepsilon_{ijk}\left(\cosh(\sqrt{g}f(P_0))J_k - \frac{g}{4}(\boldsymbol{J}.\boldsymbol{P})P_k\right) \tag{C9$'$}$$

with

$$f(P_0) \sim P_0, \text{ for small values of } P_0.$$

In this form, we see that the Poincaré kinematics corresponds to $f(P_0) = P_0$ and $g = 0$. In the case where $f(P_0) = P_0$ and $g > 0$, one obtains the Lukierski–Novicki–Ruegg κ-deformed Poincaré algebra† (κ is defined by $g = 1/\kappa^2$).

† The LNR algebra here is just the associative algebra, not the Hopf algebra.

5. The deformed Pauli–Lubanski four-vector and the deformed Casimirs

We define the deformed Pauli–Lubanski vectors as follows

$$W_0 = \boldsymbol{J}.\boldsymbol{P} \tag{25}$$

$$\boldsymbol{W} = \frac{\sinh(\sqrt{g}f(P_0))}{\sqrt{g}}\boldsymbol{J} + \boldsymbol{P}\times\boldsymbol{K}. \tag{26}$$

The 'orthogonality' relation reads

$$\frac{\sinh(\sqrt{g}f(P_0))}{\sqrt{g}}W_0 - \boldsymbol{P}\cdot\boldsymbol{W} = 0. \tag{27}$$

It is a simple matter to check that the two Casimirs are

$$C_1 = 2\frac{\cosh(\sqrt{g}f(P_0)) - 1}{g} - \boldsymbol{P}^2 \tag{28}$$

$$C_2 = \left(\cosh(\sqrt{g}f(P_0)) - \frac{g}{4}\boldsymbol{P}^2\right)W_0^2 - \boldsymbol{W}^2. \tag{29}$$

6. The problem of the position observable

The fact that the Euclidean group is part of our deformed Poincaré group has the advantage to introduce another kinematical observable which will be called the *kinematical* spin angular momentum. Such a definition is in no way related to a transformation. That is why we cannot require any rotation commutation relation *before specifying the representation we are interested in*. We only require that this kinematical observable coincides with the usual spin angular momentum for massive particles. Its definition is obtained in a way similar to the one given by Mackey [30]. Suppose that we are able to define the position observable $\boldsymbol{X}$, that is three observables X_i obeying

$$[X_i, P_j] = \mathrm{i}\delta_{ij} \tag{30}$$

the kinematical observables $\Sigma_i = J_i - \varepsilon_{ijk}X_jP_k$ obey

$$[\Sigma_1, \Sigma_2] = \mathrm{i}\Sigma_3 + \mathrm{i}\{[X_1, X_2]P_3 + [X_2, X_3]P_1 + [X_3, X_1]P_2\}P_3 \tag{31}$$

and we verify that:

- in a representation corresponding to a spinless massive particle, the condition $[X_i, X_j] = 0$ is compatible with the condition $\Sigma = 0$;
- in a representation corresponding to a spinning massive particle, when $\boldsymbol{P} = 0$, the Σ_is coincide with the spin components S_i and obey the standard commutations relations

$$[S_i, S_j] = \mathrm{i}\varepsilon_{ijk}S_k. \tag{32}$$

However, we must underline that it does not follow that the kinematical variables Σ_i obey the standard Lie algebra commutation relations.

We are left with the problem of the existence of a position observable satisfying our two requirements. It is a simple matter to check that the following definition is acceptable:

$$\boldsymbol{K} = \gamma(P_0)\boldsymbol{X}. \tag{33}$$

We note that the requirement that K_i is self-adjoint imposes some symmetrization at the RHS.

In the Galilei group, we have the relation $\boldsymbol{K} = m\boldsymbol{X}$. In the Poincaré group, this relation becomes, for spinless massive particles $\boldsymbol{K} = \frac{1}{2}[P_0, \boldsymbol{X}]_+$. Because here we are interested in the set of kinematical observables for *an arbitrary isolated system*, our choice must be independent of the representation. We note that (C7) and (C8) force the function γ of (33) to coincide with the function β. Therefore we set†

$$\boldsymbol{K} = \tfrac{1}{2}[\beta(P_0), \boldsymbol{X}]_+ \tag{34}$$

or, equivalently

$$\boldsymbol{X} = \frac{1}{2\beta(P_0)}\boldsymbol{K} + \boldsymbol{K}\frac{1}{2\beta(P_0)}. \tag{35}$$

We obtain the following commutation relations

$$[X_i, X_j] = \mathrm{i}\varepsilon_{ijk}\left(\frac{g\cosh(\sqrt{g}f(P_0))}{\sinh^2(\sqrt{g}f(P_0))}\Sigma_k - \frac{g^2}{4\sinh^2(\sqrt{g}f(P_0))}(\boldsymbol{\Sigma}.\boldsymbol{P})P_k\right) \tag{36}$$

$$[\Sigma_i P_j] = 0. \tag{37}$$

We verify easily that our definition implies that, for a spineless particle (with or without mass), the commutator $[X_i, X_j]$ vanishes.

Equation (26) becomes for the Poincaré group

$$[X_i, X_j] = -\mathrm{i}\varepsilon_{ijk}\frac{1}{P_0^2}\Sigma_k \tag{36$'$}$$

and, for the LNR quantum group (with $g = 1$)

$$[X_i, X_j] = -\mathrm{i}\varepsilon_{ijk}\left(\frac{\cosh P_0}{\sinh^2 P_0}\Sigma_k + \frac{1}{4\sinh^2 P_0}(\boldsymbol{\Sigma}.\boldsymbol{P})P_k\right). \tag{36$''$}$$

7. Conclusions

It is remarkable that the deformations we arrived at look like those of [17], where the hyperbolic functions are involved in an analogous way. In this form, the LNR appears to be the simplest of the family. In fact, this is due to our choice of parametrization. If we adopt the first presentation, namely the one with the functions α and β, this solution has no privilege. In order to underline that point, we give the first solutions of a special sequence, the one for which $\alpha(P_0) = 1 - g\,P_0^2/n^2$, with $n = 1, 2, 3, \ldots$. We get

$$\beta(P_0) = h(P_0)\left(1 - g\frac{P_0^2}{n^2}\right)^{-n/2}$$

where h is an odd polynomial of degree $2n - 1$.

† For the Poincaré group, the additivity of boosts gives rise to the interpretation of the centre of energy.

We must underline the advantage of the Poincaré group on its competitors: *we are sure that the momenta are additive, i.e. the corresponding comultiplication is co-commutative*. In particular, if an isolated system $\mathcal{S}$ is composed of two non-interacting subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$, we can say alternatively that it is composed of the system $\mathcal{S}_2$ and $\mathcal{S}_1$. There is no reason to believe that a commutative coproduct giving a bi-algebra structure to the general deformation cannot be found. From our definitions of the isolated system and the kinematical group, we are tempted to conjecture that it exists†. If it is not the case, we would have to interpret the kinematical observables attached to composite systems, a nontrivial problem which exists already for the NLR Hopf Poincaré algebra.

We note that the deformation can be chosen in such a way that Eq. (28) provides a natural cutoff for the momentum. For instance, in the LNR case, if we replace the constant κ by $i\kappa$, we get

$$P^2 = 4\kappa^2 \sin^2 \frac{P_0}{2\kappa} - m^2$$

and the fact that sinus cannot take a value larger obliges the momentum to be less than a given quantity.

Our interpretation of P_0 as the generator of time translations forbids us to introduce a renormalization which would transform the function f into the constant 1. Such a transformation would change the nature of the time axis.

The introduction of the notion of kinematical observables is a better frame than the principle defined by the author under the name of de Broglie's symmetry principle according to which all particles must be put 'on the same foot' [9]. It is clear that the kinematical observables put on the same foot all isolated systems, a statement more general than the one of elementary particles. our hope is that the position kinematical observable will help us to understand in a better way the notion of space. It seems that it is impossible to separate space, energy–momentum and spin [10]. In the usual approach, classical space and momentum space are parts of the six-dimensional phase-space (spin is ignored). In quantum mechanics, this phase-space is replaced by a non-commutative space associated with a non-trivial irreducible representation of the enveloping algebra of the Heisenberg Lie algebra. With the notion of kinematical obserable, a notion which is independent of the system, we could say that the *kinematical phase-space* is directly related to the enveloping Lie algebra of the Heisenberg Lie algebra. If we want to have spin variables involved, we have to define the *relativistic kinematical phase-space* as a non-commutative space associated with the enveloping algebra $\mathcal{H}$ of the Poincaré Lie algebra. Finally, if physics needs instead of that another Hopf algebra for a new kinematical phase space, we have to know if this last Hopf algebra is isomorphic to $\mathcal{H}$ or not.

We must recall that, in principle, experimental arguments can be used to give some restrictions to the parameter g and the function f, in the same way suggested in [18–20]. It is perhaps important to insist also on the fact that the ordinary Poincaré

† As a simple example of such an existence, take the Lie algebra defined by the bracket $[x, y] = y$; the corresponding enveloping Lie algebra has a Hopf algebra structure which is co-commutative. Define the element $z = \sinh y$ (this function has a regular inverse). It is easy to show that $[x, z] = (1 + z^2 \sinh^{-1}(z)$, a relation which defines a Hopf algebra which is obviously cocommutative. This example could be interpreted as an indication that one can construct the LNR algebra in the enveloping algebra of the Poincaré group itself. This would mean that the ordinary Poincaré algebra possesses new interpretations.

group belongs to the family, but with a non-standard interpretation, due to the definition of the position observable. However, the Poincaré group suffers a defect: it does not furnish a natural unit of length.

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